# Lecture 14 — The stability of supercurrents: Fluctuation effects

Although the discussion of BCS theory in the last 9 lectures has explained most of the microscopic properties of superconductors, we have not so far touched what is probably their most striking property, namely the stability of the current-carrying state in (e.g.) a ring geometry. Let's consider for simplicity a ring of large radius R and cross-sectional radius r, where r is not only small compared to R but also compared to the penetration depth) $\lambda$ (so that we can neglect screening effects, i.e. the vector potential (flux) provided by the circulating current) and even compared to the GL correlation length $\xi$ (so that we can neglect tranverse variations in the order parameter). We have already seen (problem 1.3) that in such a geometry, for any system of charge e, the free energy is periodic in the external flux with period h/e (not h/2e), but this is compatable with a wide variety of behavior (in particular, with "normal-metal" behavior in which F is independent of $\Phi$). To recap, a superconductor is characterized by two phenomena that are conceptually quite different: intrinsic diagmagnetism[1] (a small flux produces a corresponding circulating current, $j \propto -\text{const}\, A$), which is a thermodynamic equilibrium effect, and stability of supercurrents (a circulating current, even when it is manifestly not an equilibrium state—e.g. a nonzero current in zero external flux—doesn't decay over astronomical time scales)—a metastable phenomenon.

Before considering the phenomenon of supercurrent stability as such, it may he helpful to revisit the equilibrium diamagnetism (Meissner effect) from a more general point of view than the specific BCS theory[2]. We recall (cf. problem 1.3) that in the above geometry one way of formulating the problem of the response to an applied flux $\Phi$ is to perform a gauge transformation so as to get rid of the latter: the result is that the many-body Schrödinger equation becomes independent of $\Phi$ (i.e. the KE term is just $\sum_i p_i^2/2m$) but the single-valuedness boundary condition (SVBC) is modified in a way that depends on $\Phi$: omitting the spin degrees of freedom and the transverse orbital ones for simplicity and introducing $\theta_i$ for the angular ordinate of the i-th particle

$$\psi(\theta_1, \theta_2, ..., \theta_i + 2\pi ... \theta_N) = \exp(2\pi i \Phi/\tilde{\Phi}_0)\Psi(\theta_1, \theta_2, ...\theta_i ... \theta_N) \qquad (\tilde{\Phi}_0 \equiv h/e) \quad (1)$$

(in words: every time we take a particle once clockwise around the ring, the MBWF is multiplied by the factor $\exp(i\alpha)$, $\alpha \equiv 2\pi\Phi/\tilde{\Phi}_0$). If the groundstate energy (or at finite T the free energy) depends nontrivially on $\Phi$, we get the diamagnetic (Meissner) response; while if it is independent of $\Phi$, the system behaves "normally." For the moment, let us for simplicity restrict ourselves to the zero-temperature case.

Consider the structure of the groundstate wave function $\Psi_0(\theta_1, \theta_2 ... \theta_N)$ for $\Phi = 0$ and imagine, for the moment, taking just one particle, say $\theta_1$, once around the ring. (a) If for fixed $\theta_2 ... \theta_N$ $\Phi_0$ is exponentially localized in $\theta_1$, then we can put in the necessary "kink" in $\Psi(\Phi)$ in the region where $\Psi_0$ is exponentially small, at negligible cost in energy; thus the system behaves normally [though see below]. This is the case of a typical insulator.

[1] which in a simply connected bulk ($\gg \lambda$) geometry of course provides the Meissner effect.

[2] Refs: C.N. Yang, Revs. Mod. Phys. 34, 694 (1962); W. Kohn & D. Sherrington, RMP 42,1 (1970).

(b) If $\Psi_0$ is not localized but has nodes as a function of $\theta_1$ which are not required by symmetry, then we can put in the necessary kinks at those nodes with zero energy cost, and again the system behaves normally. The same result follows if, while $\Psi$ itself doesn't have such nodes, we can construct an alternation MBWF $\Psi_0^{'}$, which does at negligible ($O(N^{-\alpha})$) cost in energy. This is the case of a (3D) normal metal. (c) If neither of the above conditions holds, then in general it will cost a finite energy to put in the "kink," and the free energy will be a nontrivial function[3] of $\Phi$, with a periodicity that is in general the "minimum required" one, namely $h/e \equiv \tilde{\Phi}_0$.

A superconductor described by a generic BCS-type MB wave function in fact satisfies condition (a) ("single-particle localization")* and hence, the argument as developed so far would allow it to behave "normally." However, we need to go one step further and examine a process in which two particles, say for definiteness of opposite spin, are taken around the ring together. Now, the question is: Does the MBWF $\Psi(\theta_1\theta_2 : \theta_3\theta_4...\theta_N)$ vanish exponentially, or have non-symmetry-dictated nodes, as the COM coordinate $(\theta_1 + \theta_2)/2$ ranges through a large ($\sim R$) value while the relative coordinate $\theta_1 - \theta_2$ remains fixed at a small value? It is clear that if the answer is "no" for some nonzero measure set of the coordinates $\mathbf{r}_3...\mathbf{r}_N$, then that is equivalent to the statment that the 2-particle correlation function examined in problem 1.2, namely

$$K(\mathbf{r}_1\mathbf{r}_2\mathbf{r}_1'\mathbf{r}_2') \equiv \langle\psi^\dagger(\mathbf{r}_1)\psi^\dagger(\mathbf{r}_2)\psi(\mathbf{r}_1')\psi(\mathbf{r}_2')\rangle \tag{2}$$
$$\equiv \int\int d\mathbf{r}_3....d\mathbf{r}_N\Psi^*(\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3...\mathbf{r}_N)\Psi(\mathbf{r}_1'\mathbf{r}_2' : \mathbf{r}_3...\mathbf{r}_N)$$

must tend to a nonzero value in the limit examined there, namely $|\mathbf{r}_1 + \mathbf{r}_2 - \mathbf{r}_1' - \mathbf{r}_2'| \to \infty$, $|\mathbf{r}_1 - \mathbf{r}_2| \sim |\mathbf{r}_1' - \mathbf{r}_2'| \sim$ const., i.e. in the conventional language we must have ODLRO in the 2-partic1e correlation function. In this case, unless there are negligible-energy excited states that do not have this property (usually not a relevant case) we expect that the energy will, again, be a nontrivial function of $\Phi$. However, since the phase factor associated with bringing 2 particles once around the ring is $\exp 2i\alpha$ rather than $\exp i\alpha(\alpha \equiv 2\pi\Phi/\tilde{\Phi}_0)$, it is clear that the periodicity is not h/e but the smaller unit h/2e. This is exactly the case that is believed to describe superconductors (at least the classical ones).

Let's briefly note the relationship of the above argument to the more familiar one based on the Ginzburg-Landau equations. As we saw in L.10, the GL order parameter $\Psi(\mathbf{r})$ is, up to a constant which is purely a matter of convention, nothing but the quantity $\langle\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})\rangle$, i.e. the "pair wave function" $F(\mathbf{r}\mathbf{r}')$ of BCS theory evaluated at relative coordinate zero for COM coordinate $\mathbf{R} \equiv \mathbf{r}$. As we saw in problem 1.2, the appropriate limit of $K(\mathbf{r}_1\mathbf{r}_2 : \mathbf{r}_1'\mathbf{r}_2')(\mathbf{r}_1 = \mathbf{r}_2, \mathbf{r}_1' = \mathbf{r}_2', |\mathbf{r}_1 + \mathbf{r}_2 - \mathbf{r}_1' - \mathbf{r}_2'| \to \infty)$ is simply the product $\langle\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})\rangle\langle\psi_\downarrow^\dagger(\mathbf{r}')\psi_\uparrow^\dagger(\mathbf{r}')\rangle = (\text{const})\Psi(\mathbf{r})\Psi^*(\mathbf{r}')$, so the statement of ODLRO is just the statement that $\Psi(\mathbf{r})$ is finite and well defined over the whole volume of the ring. One can go through the same gauge transformation as above (though it is less usual to do so

[3]The argument as it stands is not complete, because it implicitly assumes that all relevant paths from $\theta_1$ to $\theta_1 + 2\pi$ are parallel to the $\theta_2$-axis: see AJL J. Stat. Phys 93,927(1998).

*Actually this point is very far from trivial: I hope to discuss it informally in the lecture.

in the GL formalism) and conclude that in the presence of a flux $\Phi$ the order parameter $\Psi(\theta)$ must satisfy the transformed SVBC

$$\Psi(\theta + 2\pi) = \exp 2i\alpha \cdot \Psi(\theta) \equiv \exp 2\pi i(\Phi/\Phi_0 \cdot \Psi(\theta) \qquad (\Phi_0 \equiv h/2e) \tag{3}$$

Taking into account the "bending" terms in the GL free energy then gives a nontrivial dependence of $E_0$ (or $F$) on $\Phi$, i.e. the diamagnetic (Meissner) effect.

We now turn to the question of the stability of current-carrying states, confining ourselves for simplicity for the moment to the case of zero temperature and zero external flux. The simplest way[4] to obtain such a current-carrying state in the ring geometry is simply to multiply the groundstate by the factor $\exp(il\sum_i \theta_i)(l = \text{integer})$. It is clear that the resulting wave function satisfies the SVBC and carries an angular momentum $Nl\hbar$; moreover, in the approximation of a thin and (approximately) cylindrical symmetric geometry it is (approximately) an energy eigenstate. The corresponding transformation of the GL order parameter is

$$\Psi(\mathbf{r}) = \exp(2il\theta)\Psi_0(\mathbf{r}) \tag{4}$$

(note the factor of 2!). We now examine the (meta)stability of this state.

It should be emphasized that (contrary to the impression unfortunately given in some text books) the (meta)stability of current-carrying states is by no means a trivial consequence of the diamagnetic (Meissner) effect. In fact, it is easy to construct models that show the latter feature but not the former, the simplest example being the noninteracting Bose gas[5]. We know that the current-carrying state (3) cannot be the groundstate (it is easy to show that it has an energy $N\hbar^2 l^2/2mR^2$ relative to the latter!), so the question is whether it is locally stable. At first sight, this question is simply answered within the GL formalism: Consider the GL free energy

$$F_{GL}[\Psi(\mathbf{r})] = \int d\mathbf{r}\Big\{ -\alpha|\Psi(\mathbf{r})|^2 + \frac{1}{2}\beta|\Psi|^4 + \frac{\hbar^2}{2m}|\boldsymbol{\nabla}\Psi|^2\Big\} \tag{5}$$

For the current-carrying state (1) this is $((\Psi(\mathbf{r}) = \exp(2il\theta)\Psi_0(\mathbf{r}))$

$$F_{GL} = F_0 + \frac{\hbar^2}{2m}|\Psi_0|^2 \cdot 4\ l^2/R^2 \tag{6}$$

where $\Psi_0$ and $F_0$ are respectively the OP and free energy in the thermodynamic equilibrium ($l = 0$) state.

Now consider a small deviation from this state: $\Psi(r) = \Psi(r) + \delta\Psi(r)$, and expand the energy up to second order in $|\delta\Psi(\mathbf{r})|^2$. Although it is of course possible to do a calculation for an arbitrary form of $\delta\Psi(\mathbf{r})$, it shortens the argument if we note (a) that to minimize the gradient terms it is best to mix in the groundstate, and (b) that other

[4]Though it gives only the integer-l states; half-integer values are also possible, corresponding to a factor of exp $il\theta$ on the RHS of eqn. (4)

[5]For a more general discussion of the stability of supercurrents in a dilute Bose gas, see AJL, RMP *73, 307 (2001).*

things being equal we would like to keep $\Psi(\mathbf{r})$ as close as possible to the value $(\Psi_0)$ which minimizes the "bulk" terms. This suggests that we take

$$\delta\Psi(\mathbf{r}) = -\epsilon\Psi_l(\mathbf{r}) + \delta\Psi_0(\mathbf{r}) \qquad (*) \tag{7}$$

Suppose for the moment that we could replace the space average of $|\Psi|^4$ by the squared average of $|\Psi|^2$. Then it is clear that the choice

$$\delta^2 = 2\ \epsilon - \epsilon^2 \tag{8}$$

keeps this squared average constant and thus does not affect the bulk terms in $F_{GL}$, while the gradient terms are decreased by an amount $\delta^2(4l^2/R^2)(\hbar^2/2m)\psi_0^2$. Thus under this assumption the current-carrying state would be unstable.

What saves us is the fact that any addition of the form $\delta\Psi_0(\mathbf{r})$ of a different symmetry to our original $\Psi_l(\mathbf{r})$ will make $|\Psi(\mathbf{r})|^2$ inhomogeneous in space, and therefore increase the average of $|\Psi|^4$ above the squared average of $|\Psi|^2$. In fact we have (with $\phi \equiv 2l\theta$)

$$\frac{\langle|\Psi|^4\rangle - \langle|\Psi|^2\rangle^2}{\langle|\Psi|^2\rangle^2} = \frac{\overline{|(1-\epsilon)e^{i\phi} + \delta|^4}}{|(1-\epsilon)^2 + \delta^2)|^2} - 1 \tag{9}$$

$$= \left(\frac{\overline{\left((1-\epsilon)^2 + 2\delta(1-\epsilon)\cos\phi + \delta^2\right)^2}}{\left[(1-\epsilon)^2 + \delta^2\right]^2}\right) - 1 = \frac{4\delta^2(1-\epsilon)^2\overline{cos^2}\delta}{\left[(1-\epsilon)^2 + \delta^2\right]^2}$$

$$= \frac{2\delta^2(1-\epsilon)^2}{[(1-\epsilon)^2 + \delta^2]^2} \cong 2\delta^2$$

Thus, we get an extra contribution of $\beta\delta^2\psi_0^4 = -\alpha\delta^2\psi_0^2(\alpha < 0)$ per unit volume to $F$. Assuming therefore that the best choice of $\epsilon$ is still $2\epsilon - \epsilon^2 = \delta^2$ (it is easy to see that in the limit $\delta \to 0$ one gets no advantage from any other choice) we see that the total change in free energy due to the admixture (*) is

$$\Delta F = \left(|\alpha| - \frac{\hbar^2}{2m}(4l^2/R^2)\right)\delta^2\psi_0^2 \qquad (+) \tag{10}$$

and thus is positive for sufficiently small $l$. (When the two terms become comparable, the argument can no longer be trusted quantitatively because $\epsilon$ and $\delta$ are not small; in fact, the equilibrium value of $\psi_l(\mathbf{r})$ will be depressed from $\psi_0(\mathbf{r})$ accordingly to the arguments of L.10. It is nevertheless interesting that when interpreted in terms of the superfluid velocity $\mathbf{v}_s \equiv \hbar/2m\boldsymbol{\nabla}(\arg\Psi(\mathbf{r}))$ equation (+) gives a critical velocity equal to $\hbar/2m\xi$, which differs from the true value $3^{-1/2}(\hbar/2m\xi)$ (L.10) only by a numerical factor $\sim 1$).

It is interesting to look at this question more explicitly in terms of the topology of the GL OP (or equivalently of the MBWF). We recall that we define the superfluid velocity $\mathbf{v}_s$ by

$$\mathbf{v}_s = (\hbar/2m)\boldsymbol{\nabla}\phi \tag{11}$$

where $\phi$ is the phase of the GL order parameter (the COM wave function of the Cooper pairs). Since this phase must be single-valued modulo $2\pi$, we have for any closed contour the Onsager-Feynman quantization rule

$$\oint \mathbf{v}_s \cdot d\mathbf{l} = nh/2m \qquad\qquad n = 0, \pm 1, \pm 2..... \tag{12}$$

In a ring geometry, the integer $n$ counts the "number of turns" made by the (complex) order parameter around the origin in the Argand plane as we go once around the ring; it is therefore often called the "winding number." Now it is intuitively clear that to change the winding number the path in the Argand plane must be deformed so as to cut the origin, which means, to the extent that we maintain the OP to be a function only of $\theta$ and independent of the transverse directions, that it must go to zero over some cross-section of the ring. The situation so far is no different from that of an electron in an excited ($l \neq 0$) state in an atom[6]; to return to the groundstate such an electron also has to change its winding number, and this again requires a node to occur, at some point in the process, across some surface intersecting the nucleus. So why is this process quite straightforward for the electron but all but impossible for the Cooper pairs? The answer is that the electron obeys a simple linear time-dependent Schrödinger equation, and therefore it takes no extra energy to create a node; in fact, if for definiteness we start in a $p$-state and wish to return to $s$-state (groundstate) then it is easy to see that the linear combination

$$\begin{aligned} \psi(\mathbf{r}:t) = a_p(t)\psi_p(\mathbf{r}) + a_s\psi_s(\mathbf{r}) \\ (|a_s(t)|^2 + |a_p(t)|^2 = 1 \end{aligned} \tag{13}$$

where $a_s(t)$ increases monotonically from 0 to 1, has an energy (up to a constant)

$$\begin{aligned} E(t) = E_p|a_p(t)|^2 + E_s|a_s(t)|^2 \\ = const. + \frac{1}{2}\big(E_p - E_s\big)\big(|a_p(t)|^2 - |a_s(t)|^2\big) \end{aligned} \tag{14}$$

which thus decreases monotonically with time as the transition proceeds; this is a simple consequence of the linearity of the Schrödinger equation (Note that equation (13) indeed implies the existence of a nodal surface at the time when $|a_p| = |a_s|$). By contrast, the GL equation is not linear, and it is fairly easy to see that the $|\Psi(\mathbf{r})|^4$ term (which we recall is always associated with a positive coefficient) means that forming a node costs extra energy: e.g. if such a term were present in the atomic problem, there would be an extra term proportional to $|a_p(t)|^2 \cdot |a_s(t)|^2$ with a positive coefficient. In other words, there is in general a free energy barrier between states of different winding number, and it is this feature which lies at the root of the metastability of supercurrents.

At this point it becomes an obvious question whether this free energy barrier might be overcome, as in other similar cases, by thermal and possibly quantum fluctuations, and

[6] In the semiclassical approximation in which the electromagnetic field is treated classically ("induced" transition).

we will shortly go on to discuss the question. However, before doing so it is important to note that the above argument, which is essentially the standard textbook one (see e.g. Tinkham section 8.5), contains a serious lacuna. The point is that it is implicitly based on the assumption of a unique GL order parameter, i.e. on the assumption that all the $N$ electrons in the system are paired in the, same two-particle ("quasimolecular") state. But it is not at all clear that this need be so; for example, if we start off with all $N/2$ Cooper pairs in a state with finite winding number $n = l$, and wish to return to the groundstate ($n = 0$), nothing prevents us from gradually transferring pairs from the one state to the other, so that at an intermediate stage we have $(N - M)/2$ in one and $M/2$ in the other, i.e., apart from normalization

$$\Psi = (b_l^\dagger)^{N-M/2}(b_0^\dagger)^{M/2}|vac > \equiv \Psi_F^{(M)} \tag{15}$$

where $b_0^\dagger$ and $b_l^\dagger$ create a Cooper pair in the $n = 0$ and $n = l$ states respectively. (Formally, we have $b_0^\dagger \equiv \sum_k c_k a_{k\uparrow}^\dagger a_{-k\downarrow}^\dagger$, $b_l^\dagger \equiv \sum_k c_k a_{l/2R+k,\uparrow}^\dagger a_{l/2R-k,\downarrow}^\dagger$). Let us call states of the type (15) "Fock" states: note that they are different from "coherent" states in which all $N/2$ pairs are condensed into a single linear combination of the $l$ and 0 states, and which corresponds to a unique GL order parameter,

$$\Psi_{coh} \equiv (a_l b_l^\dagger + a_0 b_0^\dagger)^{N/2}|vac > \tag{16}$$

So the question arises: Is it obvious that by tracing out a path going through states of the type (15) ("Fock path") rather than (16) ("coherent path"), the system could not as it were evade the free energy barrier which we have seen occurs on the "coherent" path?

One possible approach is to note that although the states (15) and (16) are formally different, there is a close relationship between them. In fact, let us for simplicity consider the case $M = N/2$ and consider a generalized version of state (16), namely

$$\Psi_{coh} \equiv \left(2^{-1/2}(b_l^\dagger + b_0^\dagger e^{i\phi})\right)^{N/2}|vac > \tag{17}$$

Then it is easy to show that the Fock state $\Psi_F(N/2)$ is just a superposition of the coherent states (16):

$$\Psi_F^{N/2} = (2\pi)^{-1}\int_0^{2\pi} d\phi \Psi_{coh}(\phi)\exp -i(N/4)\phi \tag{18}$$

From equation (7) it is at least, intuitively plausible that the free energy barrier on the Fock path cannot be less than that on the "best" coherent path[7], and this indeed turns out to be true (this is rather obvious to the extent that the Hamiltonian is effectively diagonal in the $\phi$-representation, which is always true in the thermodynamic limit). Another way of looking at this result is by comparison with the two-particle problem.

[7]For a cylindrically symmetric geometry the free energy barrier on a coherent path is independent of $\phi$, but for the more general case this need not be so, see AJL, loc. cit.

Suppose we have two particles interacting in free space with some attractive potential $V(\mathbf{r})$ whose Fourier transform is $V_{\mathbf{k}}$, so that the groundstate is bound, i.e., a "molecule." If the COM is at rest, then the two-body wave function is of the form (omitting the spin suffixes for simplicity)

$$\Psi_0 = \sum_{\mathbf{k}} c_{\mathbf{k}} a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} |vac> \tag{19}$$

with the normalization constraint

$$\sum_{\mathbf{k}} |c_{\mathbf{k}}|^2 = 1. \tag{20}$$

The energy is just the molecular binding energy, let us say $E_b$. Similarly, by Galilean invariance, the wave function for the molecule moving with COM momentum $\hbar K$ is

$$\Psi_{\mathbf{K}} = \sum_{\mathbf{k}} c_{\mathbf{k}} a^\dagger_{\mathbf{K}/2+\mathbf{k}} a^\dagger_{\mathbf{K}/2-\mathbf{k}} |vac> \tag{21}$$

The corresponding energy is $E_b + \hbar^2 K^2/2(2m)$.

Now, we can of course consider (a) a linear superposition of the form

$$\Psi = a\Psi_K + b\Psi_0, \qquad |a|^2 \equiv p_1, \qquad |b|^2 \equiv p_2, \qquad p_1 + p_2 = 1 \tag{22}$$

(b) an incoherent mixture of $\Psi_K$ and $\Psi_0$ with weights $p_1$ and $p_2$ respectively. Because of the linearity of the Schrödinger equation and the conservation of momentum, the energy of these two states is identical and equal to $-E_b + p_1\hbar^2 K^2/2(2m)$: it is therefore a monotonically increasing function of $p_1$ (just as for the case of the electron in the atom).

A closer analog to the many-body state (15) is obtained if we consider a different possibility: Suppose we arbitrarily split the wave vectors $\mathbf{k}$ into two groups 1 and 2 such that

$$\sum_{\mathbf{k}\epsilon 1} |c_{\mathbf{k}}|^2 = p_1, \qquad \sum_{\mathbf{k}\epsilon 2} |c_{\mathbf{k}}|^2 = p_2 \qquad , p_1 + p_2 = 1 \tag{23}$$

The assignation of a particular $k$ to group 1 or 2 is taken to be "random": specifically, any correlations die out over a range that in energy space is much less than $E_b$. Consider now a state of the form

$$\Psi_{coh} = \left( \sum_{\mathbf{k}\epsilon 1} c_{\mathbf{k}} a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} + \sum_{\mathbf{k}\epsilon 2} c_{\mathbf{k}} a^\dagger_{\mathbf{K}/2+\mathbf{k}} a^\dagger_{\mathbf{K}/2-\mathbf{k}} \right) |vac> \tag{24}$$

This state is normalized, and it is easy to see that the kinetic energy, both COM and relative, is the same as for the state (22). However, the potential energy (a two-body operator) is different:

Consider first the state (22). In this case, it is clear that there is no interference between the two terms, since they correspond to different COM momentum (i.e,

$\mathbf{k}$, $-\mathbf{k}$, $\not\Rightarrow$ $\mathbf{K}/2+\mathbf{k}', \mathbf{K}/2-\mathbf{k}'$ from momentum conservation). The potential energy is therefore

$$\langle V\rangle = p_1 \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} c_{\mathbf{k}} c^*_{\mathbf{k}'} + p_2 \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}/2-\mathbf{k}',\mathbf{K}/2+\mathbf{k}'} c_{\mathbf{k}} c^*_{\mathbf{k}'} \tag{25}$$

where the sum runs over all $\mathbf{k}$. Since $V$ is a function only of $\mathbf{k}-\mathbf{k}'$, this reduces to the expression

$$\langle V\rangle = (p_1+p_2)\sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} c_{\mathbf{k}} c^*_{\mathbf{k}'} = \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} c_{\mathbf{k}} c^*_{\mathbf{k}'} \tag{26}$$

In the state (23), by contrast, we still get no interference between the $\mathbf{K}=0$ and $\mathbf{K}\neq 0$ terms, but within each term the potential is reduced:

$$\langle V\rangle = \sum_{\mathbf{k},\mathbf{k}'\epsilon 1} V_{\mathbf{k}\mathbf{k}'} c_{\mathbf{k}} c^*_{\mathbf{k}'} + \sum_{\mathbf{k},\mathbf{k}'\epsilon 2} V_{\mathbf{k}\mathbf{k}'} c_{\mathbf{k}} c^*_{\mathbf{k}'} \tag{27}$$

We have lost the terms for which $\mathbf{k}$ is in group 1 and $\mathbf{k}'$ in group 2! If we assume, in accordance with the "random" prescription for assigning $\mathbf{k}$, that the "local" as well as global weight of this group is proportional to $p_1 p_2$ then we find that the resulting contribution to the energy is $-\langle V\rangle_0 p_1 p_2$, where the negative quantity $\langle V\rangle_0$ is the expectation value in the simple molecular state. Thus, for the class of states (23) we have for the total energy as a function of $p_1$, $p_2$:

$$\begin{aligned} E &= -E_b + \big(\hbar^2 K^2/2(2m)\big)p_1 + |\langle V\rangle|_0 p_1 p_2 \\ &\equiv -E_b + \big(\hbar^2 K^2/2(2m)\big)p_1 + |\langle V\rangle|_0 p_1(1-p_1) \end{aligned} \tag{28}$$

This expression is not necessarily monotonic as a function of $p_1$, in fact, if $|\langle V\rangle|_0 > \hbar^2 K^2/2(2m)$ it has a maximum at $p_1 < 1$. It is easy to see that exactly the same conclusions follow for a mixture of the states (23). Although this construction is highly artificial for the two-body problem, it may give some insight into the way in which metastability arises in the many-body case[8],[9]

Let us now turn to the question of fluctuations and in particular their role in limiting the prima facie infinite metastability of the supercurrent. I will anticipate the result that, in almost any context except possibly the very smallest samples currently obtainable, the effect of fluctuations is negligible the moment one gets appreciably away from $T_c$, and thus the consideration of such effects can legitimately be done entirely within the GL framework.

The basic principle involved is very simple: The GL free energy $F$ is a functional $F\{\Psi(\mathbf{r})\}$ of the OP $\Psi(\mathbf{r})$, and the usual GL equation is obtained by finding a (global or local) minimum of $F$. : call the relevant value $F_0$. If one now wishes to consider a nonequilibrium configuration of $\Psi(\mathbf{r})$, i.e., a fluctuation, one needs to assign to it a

[8] If (23) in the two-body case is the analog of (15) in the many-body case, one might ask what many-body state is the analog of the two-body state (22)? The answer is the "Schrödinger-cat"-like state $\big(a_l(b_l^\dagger)^{N/2} + a_0(b_0^\dagger)^{N/2}\big)|vac>$.

[9] Explicit evaluation of the energies for the ansatz (15) is complicated by the fact that $b_o^+$ and $b_l^+$ may occupy the same single-particle state.

probability[10] $Z^{-1} \exp -\beta \Delta F\{\Psi(\mathbf{r})\}$ where $\Delta F\{\Psi(\mathbf{r})\} \equiv F\{\Psi(\mathbf{r})\} - F_0(> 0)$ and Z is the partition function, which is formally defined by the functional integral

$$Z = \int^{\infty} \mathcal{D}\Psi(\mathbf{r}) \exp -\beta \Delta F\{\Psi(\mathbf{r})\} \tag{29}$$

thus giving a total probability of 1 for all possible fluctuations. We are particularly interested in the probability of a fluctuation that takes us to the lowest saddle-point in the free energy barrier that separates states of different winding number. (It is a natural and reasonable assumption that once the system has "climbed up" to the saddlepoint it will have no difficulty "rolling down" to the deeper minimum on the far side). What is the nature of this saddlepoint? For the thin ($\ll \xi, \lambda$ wire we are considering, it is clear that it is not optimal to depress the OP to zero uniformly over the circumference of the loop: rather, it is energetically advantangeous to do so over a small region, and by balancing off the bulk and (extra) kinetic energies against one another we see that the optimal length of the region should be of order of the GL healing (correlation) length $\xi(T)$. The free energy $\Delta F$ required is therefore of the order $\mathcal{F}_0(T)\xi(T)A$ where $A$ is the cross-section of the wire and $\mathcal{F}_0(T)$ is the superconducting condensation energy.

A more detailed calculation by Langer and Ambegaokar confirms this result, with a numerical factor $8\sqrt{2}/3$ for current $\ll I_c$:

$$\Delta F = 8\sqrt{2}/3 \mathcal{F}_0(T)\xi_0(T)A \tag{30}$$

Two points that are not immediately obvious but follow from the LA calculation are:

(1) the saddlepoint does not occur at the point where $\Psi(\mathbf{r})$ goes through zero across some cross-section, but on the "climb up" to it;

(2) (a related point): the saddlepoint for the transition from $n = l$ to $n = l - 1$ is lower than that to lower values of $l$, in particular to $l = 0$. Thus the phase slips $2\pi$ at a time, not more. (This is opposite to the conclusion of our earlier, naive argument).

It is convenient to write the Gibbs exponent $\Delta F/kT$ for the fluctuation in the form (approximating $T$ by $T_c$ in the denominator).

$$\Delta F = const \cdot (1 - T/T_c)^{3/2} F_0 \, \xi_0^3 \, (A/\xi_0^2)/kT_c \tag{31}$$

where $F_0$ is the free energy at zero temperature and $\xi_0$ the (zero-temperature) Pippard coherence length. The crucial point to bear in mind is that the dimensionless number $\eta \equiv F_0 \xi_0^3/kT_c$ is typically quite enormous: in fact, using $F_0 \sim (kT_c)dn/d\epsilon \sim (kT_c)^2 n/\epsilon_F$, we see that $\eta$ is of order $(kT_c/\epsilon_F) \cdot n\xi_0^3$, which typically in a classic superconductor is of order $10^{-4} \cdot 10^{12} \sim 10^8$. Thus, unless A is very small compared to $\xi_0^2$, phase slips can occur with reasonable probability only very close to $T_c$.

[10] Note that the thermal energy which governs the fluctuation of the ("macroscopic") order parameter is simply $\beta^{-1} \equiv k_B T$ not $\sim N k_B T$! (The order parameter represents a single degree of freedom).

For small currents compared to the critical current $I_c(T)$ the above analysis, done explicitly for a ring in zero flux, applies equally well (as shown by LA) to a thin wire biased by a constant current. In the latter situation the phase slips have to be compensated by a "winding up" of the phase difference across the ends, so that

$$(d\phi/dt)_{phaseslip} = (d\phi/dt)_{ends} = 2eV/\hbar \tag{32}$$

Although the rate of crossing $l \to l-1$ is independent of the current, when we subtract out the reverse rate $l-1 \to l$ the net result is proportional to $I$ for small $I$ (this is because the height of the barrier relative to the starting state is slightly higher for the reverse transitions by an amount proportional to I.) Thus the phase slips generate a linear (ohmic) resistance. At larger I, the I-V characteristic will in general be nonlinear.

For thick wires, not too close to $T_c$, the phase slip rate for $I \ll I_c(T)$ is so slow as to be unobservable, and to see anything interesting one has to go to currents $I \sim I_c(T)$ (or superfluid velocities $v_s \sim v_c(T)$): as $I$ (or $v_s$) increases, the equilibrium OP, and hence the free energy necessary to turn a cross-section normal, decreases as analyzed in L.10 and as a result phase slips become easier. It is now necessary to distinguish the case of a ring in zero flux (where the superfluid velocity is limited by the critical value $v_c(T)$) and that of a wire biased by an external current (where the current is limited by the critical value $I_c(T)$); remember that maximum $I_c$ does not correspond to maximum $v_s$! In either case the effective free energy barrier decreases to zero as a power of $\epsilon \equiv 1 - I/I_c(T))$ (etc.), but further analysis is needed to determine what that power is (for the current-biased wire LA show that it is 1/2). Thus, in any wire of finite thickness, at any nonzero temperature, one expects in principle to see a "rounding" of the abrupt superconducting transition due to phase slips, but the width of this rounding is often unobservably small.

The very large value of the parameter $\eta$ in the classic superconductors has the consequence, quite generally, that appreciable fluctuations of the order parameter averaged over a coherence length (or healing length) are negligible except extremely close to $T_c$. This is the fundamental reason why BCS theory, which is essentially a kind of mean-field theory, works so spectacularly well[11]. Of course, the deviations due to fluctuation effects (such as the extra "Aslamazov-Larkin" conductivity just above $T_c$) are of considerable interest: see Tinkham section 6.

Two final notes:

(1) for type-II superconductors in fields $> H_{c1}$, situation is more complicated because pre-existing vortices can move across current and thereby decrease it.

(2)possibility of finite resistance due to "quantum tunnelling" phase slips at low temperatures.

[11] cf. problem 2 on PS 3